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A simple approach to the Cauchy problem for complex Ginzburg–Landau equations by compactness methods

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ABSTRACT

This paper is concerned with the Cauchy problem (CGL) in $L^2(\mathbb{R}^N)$ for complex Ginzburg–Landau equations with Laplacian Δ and nonlinear term $|u|^{q-2}u$ multiplied by the complex coefficients $\lambda + i\alpha$ and $\kappa + i\beta$, respectively ($q \geqslant 2$, $\lambda > 0$, $\kappa > 0$, $\alpha, \beta \in \mathbb{R}$). The global existence of strong solutions to (CGL) is established without any upper restriction on $q \geqslant 2$ but with some restriction on α/λ and β/κ. The result corresponds to Ginibre and Velo (1996) [3, Proposition 5.1] which is technically proved by combining convolution (regularizing) methods with compactness (localizing) methods, while our proof here is fairly simplified. The key to our proof is the Cauchy problem $(\mathrm{CGL})_R$ which is (CGL) with Δ replaced with $\Delta - V_R$, where $V_R(x) := (|x| - R)^2$ $(|x| > R)$, $V_R(x) := 0$ $(|x| \leqslant R)$. The solvability of $(\mathrm{CGL})_R$ is a direct consequence of Okazawa and Yokota (2002) [16, Theorem 4.1]. Taking the limit of global strong solutions to $(\mathrm{CGL})_R$ as $R \to \infty$ yields a global strong solution to (CGL). The result gives also an unbounded version of Okazawa and Yokota (2002) [16, Theorem 1.1 with $p = 2$] for the initial–boundary value problem on bounded domains.

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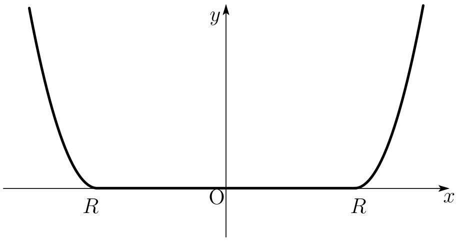

Fig. 1. $y = V_R(x)$ $(N = 1)$.

1. Introduction and results

The complex Ginzburg–Landau equation, originally derived by Newell–Whitehead [10], appears in the mathematical description of spatial pattern formation and of the onset of instabilities in nonequilibrium fluid dynamical systems (see Cross–Hohenberg [2] and Mielke [9]). The first basic mathematical problem in the study of the complex Ginzburg–Landau equation is to prove the *existence and uniqueness of solutions* to the initial–boundary value or Cauchy problem (for the initial–boundary value problem see e.g., Temam [20], Okazawa–Yokota [14–17], Ogawa–Yokota [11] and Matsumoto–Tanaka [6–8]; for the Cauchy problem see e.g., Yang [21], Ginibre–Velo [3] and [4]).

The first topic of this paper is to discuss the solvability of the following Cauchy problem for the complex Ginzburg–Landau *type* equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\lambda + i\alpha)(-\Delta + V_R)u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{CGL_R}$$

Here $u = u(x,t)$ is a complex-valued unknown function, and

$$\lambda > 0, \qquad \kappa > 0, \qquad \alpha, \beta, \gamma \in \mathbb{R}, \qquad q \geqslant 2, \qquad N \in \mathbb{N} \tag{1.1}$$

are constants. The initial data u_0 satisfies

$$u_0 \in H^1(\mathbb{R}^N) \cap D(V_R^{1/2}) \cap L^q(\mathbb{R}^N), \quad R \geqslant 0, \tag{1.2}$$

where $D(V_R^{1/2}) := \{u \in L^2(\mathbb{R}^N);\ V_R^{1/2}u \in L^2(\mathbb{R}^N)\}$ is the Hilbert space with inner product

$$(u,v)_{D(V_R^{1/2})} := (u,v)_{L^2} + \big(V_R^{1/2}u, V_R^{1/2}v\big)_{L^2}$$

and $V_R \in C^1(\mathbb{R}^N; [0,\infty))$ is defined as

$$V_R(x) := \begin{cases} (|x| - R)^2 & \text{if } |x| > R, \\ 0 & \text{if } |x| \leqslant R. \end{cases} \tag{1.3}$$

Note that $V_0(x)$ coincides with the usual harmonic oscillator $|x|^2$ and $\lim_{R\to\infty} V_R(x) = 0$ for each $x \in \mathbb{R}^N$ (see Fig. 1). Therefore $(\mathrm{CGL})_\infty$ is the Cauchy problem for the usual complex Ginzburg–Landau equation. Since $(-\Delta + V_R)^{-1/2}$ is compact in $L^2(\mathbb{R}^N)$ (see Lemma 3.2 below), we can apply compactness methods directly to $(\mathrm{CGL})_R$. Indeed, as a direct application of [16, Theorem 4.1], we can show the

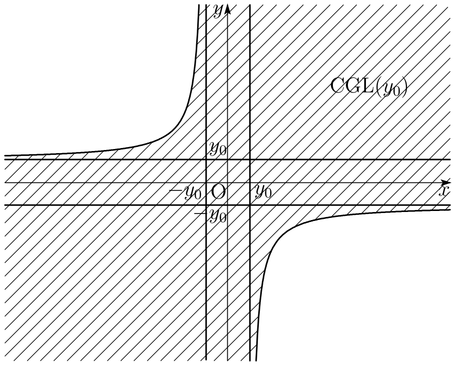

Fig. 2. The boundary of $\mathrm{CGL}(y_0)$ is given by a pair of hyperbolas.

existence of strong solutions to $(\mathrm{CGL})_R$ (in the sense of Definition 1.1) without any upper restriction on $q \geqslant 2$ but with the following restriction on α/λ and β/κ:

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in \mathrm{CGL}\left(\frac{2\sqrt{q-1}}{q-2}\right) \tag{1.4}$$

(see Theorem 1.1 below). Here $\mathrm{CGL}(y_0)$ $(0 < y_0 < \infty)$ is defined as

$$\mathrm{CGL}(y_0) := \left\{(x, y) \in \mathbb{R}^2;\ xy \geqslant 0 \text{ or } \frac{|xy| - 1}{|x| + |y|} < y_0\right\} \tag{1.5}$$

(see Fig. 2) and $\mathrm{CGL}(\infty) := \mathbb{R}^2$.

The second topic is to take the limit of strong solutions to $(\mathrm{CGL})_R$ as $R \to \infty$. Namely, we show the existence of strong solutions to the following Cauchy problem for the *usual* complex Ginzburg–Landau equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{CGL}$$

where $\lambda, \kappa, \alpha, \beta, \gamma, q, N$ satisfy (1.1) and u_0 satisfies

$$u_0 \in H^1\big(\mathbb{R}^N\big) \cap L^q\big(\mathbb{R}^N\big). \tag{1.6}$$

Our result (see Theorem 1.2 below) corresponds to [3, Proposition 5.1] which is technically proved by combining convolution (regularizing) methods with compactness (localizing) methods, while our approach here is fairly simpler. Once the solvability of $(\mathrm{CGL})_R$ is established as stated above, the limit of strong solutions to $(\mathrm{CGL})_R$ as $R \to \infty$ yields a strong solution to (CGL). The result gives also an unbounded version of [16, Theorem 1.1 with $p = 2$] for the initial–boundary value problem on bounded domains.

To state our results we give the definition of strong solutions to $(\mathrm{CGL})_R$ (and (CGL)).

Definition 1.1. A function $u(\cdot) \in C([0, \infty); L^2(\mathbb{R}^N))$ is called a *strong solution* to $(\mathrm{CGL})_R$ if $u(\cdot)$ has the following three properties:

(a) $u(t) \in H^2(\mathbb{R}^N) \cap L^{2(q-1)}(\mathbb{R}^N)$, $V_R u(t) \in L^2(\mathbb{R}^N)$ a.a. $t > 0$;

(b) $(\partial u/\partial t)(\cdot)$, $\Delta u(\cdot)$, $V_R u(\cdot)$, $|u|^{q-2}u(\cdot) \in L^2(0,T;L^2(\mathbb{R}^N))$ for every $T>0$;
(c) $u(\cdot)$ satisfies the equation in $(\mathrm{CGL})_R$ a.e. on $(0,\infty)$ as well as the initial condition.

Also, a function $u(\cdot) \in C([0,\infty);L^2(\mathbb{R}^N))$ is said to be a *strong solution* to (CGL) if $u(\cdot)$ has the three properties (a)–(c) with V_R replaced with 0. In view of (b) we see that if $u(\cdot)$ is a strong solution to $(\mathrm{CGL})_R$ (or (CGL)), then $u(\cdot) \in H^1(0,T;L^2(\mathbb{R}^N)) \subset C^{0,1/2}([0,T];L^2(\mathbb{R}^N))$ for every $T>0$.

Now we state our results on the above two topics.

The first theorem is concerned with the solvability of $(\mathrm{CGL})_R$, i.e., the existence of strong solutions to $(\mathrm{CGL})_R$ with $u_0 \in H^1(\mathbb{R}^N) \cap D(V_R^{1/2}) \cap L^q(\mathbb{R}^N)$.

Theorem 1.1. *Let* (1.1) *be satisfied. Let u_0 be as in* (1.2)*. Assume that $(\alpha/\lambda, \beta/\kappa)$ satisfies* (1.4)*. Then for all $R \geqslant 0$ there exists a strong solution $u(\cdot) \in C([0,\infty);L^2(\mathbb{R}^N))$ to* $(\mathrm{CGL})_R$ *such that*

$$u(\cdot) \in C\big([0,\infty); H^1\big(\mathbb{R}^N\big) \cap D\big(V_R^{1/2}\big) \cap L^q\big(\mathbb{R}^N\big)\big), \tag{1.7}$$

and for every $t>0$,

$$\big\|u(t)\big\|_{L^2} \leqslant e^{\gamma t}\|u_0\|_{L^2}, \tag{1.8}$$

$$E_R\big(u(t)\big) + \eta \int_0^t \big\{\delta^2 \big\|(-\Delta + V_R)u(s)\big\|_{L^2}^2 + \big\|u(s)\big\|_{L^{2(q-1)}}^{2(q-1)}\big\}\,ds \leqslant e^{\gamma_+ qt} E_R(u_0), \tag{1.9}$$

where $\gamma_+ := \max\{\gamma, 0\}$,

$$E_R(u) := \frac{\delta^2}{2}\big[\|\nabla u\|_{L^2}^2 + \big\|V_R^{1/2}u\big\|_{L^2}^2\big] + \frac{1}{q}\|u\|_{L^q}^q,$$

and $\delta>0$, $\eta>0$ are constants depending only on $\lambda, \kappa, \alpha, \beta, q$.

The second theorem is concerned with the solvability of (CGL), i.e., the existence of strong solutions to (CGL) with $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.

Theorem 1.2. *Let* (1.1) *be satisfied. Let u_0 be as in* (1.6)*. Assume that $(\alpha/\lambda, \beta/\kappa)$ satisfies* (1.4)*. Then there exists a strong solution $u(\cdot) \in C([0,\infty);L^2(\mathbb{R}^N))$ to* (CGL) *such that*

$$u(\cdot) \in C\big([0,\infty); H^1\big(\mathbb{R}^N\big) \cap L^q\big(\mathbb{R}^N\big)\big), \tag{1.10}$$

and for every $t>0$, (1.8) *and the following estimate hold:*

$$E_\infty\big(u(t)\big) + \eta \int_0^t \big\{\delta^2 \big\|\Delta u(s)\big\|_{L^2}^2 + \big\|u(s)\big\|_{L^{2(q-1)}}^{2(q-1)}\big\}\,ds \leqslant e^{\gamma_+ qt} E_\infty(u_0), \tag{1.11}$$

where

$$E_\infty(u) := \frac{\delta^2}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{q}\|u\|_{L^q}^q,$$

and γ_+, δ, η are the same constants as in Theorem 1.1.

Finally, we give the *uniqueness* of strong solutions to $(\mathrm{CGL})_R$ and (CGL) by the same argument as in the proof of [17, Theorem 1.2] (cf. [3, Proposition 4.2]) under the additional condition that

$$2 \leqslant q < 2^* := \begin{cases} 2 + \frac{4}{N-2} & (N \geqslant 3), \\ \infty & (N = 1, 2). \end{cases} \tag{1.12}$$

Consequently, we can obtain the following existence and uniqueness theorem.

Theorem 1.3. *Let $R \geqslant 0$. Let (1.1) be satisfied. Let u_0 be as in (1.2) (or (1.6)). Assume that $(\alpha/\lambda, \beta/\kappa)$ satisfies (1.4). Assume further that q satisfies (1.12). Then there exists a unique strong solution $u(\cdot) \in C([0,\infty); L^2(\mathbb{R}^N))$ to $(\mathrm{CGL})_R$ (or (CGL)). Moreover, let $u(\cdot)$ and $v(\cdot)$ be strong solutions to $(\mathrm{CGL})_R$ (or (CGL)) with initial data u_0, v_0 satisfying (1.2) (or (1.6)), respectively, and put $w(\cdot) := u(\cdot) - v(\cdot)$ and $w_0 := u_0 - v_0$. Then for every $t > 0$,*

$$\|w(t)\|_{L^2}^2 + \lambda \int_0^t e^{\int_s^t K(r)\,dr} \left\{ \|\nabla w(s)\|_{L^2}^2 + \|V_R^{1/2} w(s)\|_{L^2}^2 \right\} ds \leqslant e^{\int_0^t K(r)\,dr} \|w_0\|_{L^2}^2 \tag{1.13}$$

(or (1.13) with $V_R \equiv 0$), where $K(\cdot)$ is a continuous function depending only on $\lambda, \kappa, \beta, \gamma, q$ and $E_R(u_0)$, $E_R(v_0)$ (or $E_\infty(u_0)$, $E_\infty(v_0)$).

The advantage of our method here is that the unbounded potential V_R enables us to apply our previous theorem [16, Theorem 4.1] to $(\mathrm{CGL})_R$ and derive the existence result for (CGL) by a simple limiting procedure as $R \to \infty$. Indeed, [16, Theorem 4.1] asserts the solvability for abstract evolution equations with subdifferential operators in a complex Hilbert space of the form:

$$\frac{du}{dt} + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \tag{A-CGL}$$

where $\lambda > 0$, $\kappa > 0$ and α, β, $\gamma \in \mathbb{R}$ are constants. Note that $\partial\varphi$ is a nonlinear operator in general. From this point of view we expect to obtain the solvability of (CGL) with Δ replaced with the p-Laplacian Δ_p:

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right), \quad p > 2.$$

Concerning this problem, one cannot apply the semilinear theory, such as contraction methods for the corresponding integral equation. Therefore our method would play an essential role in such problem. We will discuss this topic in our forthcoming paper.

This paper is organized as follows. In Section 2 we consider the Cauchy problem for the abstract evolution equation (A-CGL) as stated above and review an abstract theorem established in [16] which will be applied directly to $(\mathrm{CGL})_R$. Section 3 is devoted to the proof of Theorem 1.1 (the solvability of $(\mathrm{CGL})_R$) which is accomplished by verifying conditions in the abstract theorem. In Section 4 we prove Theorem 1.2 (the solvability of (CGL)) by letting $R \to \infty$ in $(\mathrm{CGL})_R$. Finally, Theorem 1.3 (the uniqueness of solutions) is proved in Section 5.

2. Compactness methods for (A-CGL)

Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. First we review the definition of subdifferential operators in X. Let $\varphi : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function, where proper means that the effective domain $D(\varphi) := \{u \in X;\ \varphi(u) < \infty\}$ is nonempty. Then the subdifferential of φ at $u \in D(\varphi)$ is defined as the set of all $f \in X$ such that $\mathrm{Re}(f, v - u) \leqslant \varphi(v) - \varphi(u)$ for every $v \in X$, and denoted by $\partial\varphi(u)$. It is well-known that $\partial\varphi$ is a (possibly multivalued) m-accretive operator in X and the Yosida approximation $(\partial\varphi)_\varepsilon$ of $\partial\varphi$ is defined as

$$(\partial\varphi)_\varepsilon := \frac{1}{\varepsilon}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon\partial\varphi)^{-1},\ \varepsilon > 0,$$

which coincides with the Fréchet derivative of the Moreau–Yosida regularization φ_ε:

$$\varphi_\varepsilon(v) := \min_{w\in X}\left\{\varphi(w) + \frac{1}{2\varepsilon}\|w - v\|^2\right\}, \quad v \in X,\ \varepsilon > 0$$

(see Brezis [1, Proposition 2.11]). So we use the simplified notation $\partial\psi_\varepsilon := (\partial\psi)_\varepsilon$.

Next let φ, $\psi : X \to [0,\infty]$ be proper lower semi-continuous convex functions on X. Assume that $\partial\varphi$, $\partial\psi$ are single-valued. Then we formulate the abstract Cauchy problem for Eq. (A-CGL) introduced in Section 1:

$$\begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)\partial\varphi(u) + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{cases} \tag{ACP}$$

where $\lambda > 0$, $\kappa > 0$, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants and u is an X-valued unknown function. In the same way as in [16, Section 4] we shall employ the following conditions on φ, ψ:

(**A1**) The sublevel set $\{u \in D(\varphi);\ \varphi(u) \leqslant c\}$ is compact in X for each $c > 0$.
(**A2**) $\exists p \in [2,\infty)$ such that $\varphi(\zeta u) = |\zeta|^p\varphi(u)$, $u \in D(\varphi)$, $\zeta \in \mathbb{C}$, $\operatorname{Re}\zeta > 0$.
(**A3**) $\exists q \in [2,\infty)$ such that $\psi(\zeta u) = |\zeta|^q\psi(u)$, $u \in D(\psi)$, $\zeta \in \mathbb{C}$, $\operatorname{Re}\zeta > 0$.
(**A4**) $\exists c_p \geqslant 0$ such that for $u, v \in D(\partial\varphi)$,

$$\left|\operatorname{Im}\big(\partial\varphi(u) - \partial\varphi(v), u - v\big)\right| \leqslant c_p \operatorname{Re}\big(\partial\varphi(u) - \partial\varphi(v), u - v\big).$$

(**A5**) $\exists c_q \geqslant 0$ such that for $u \in D(\partial\varphi)$ and $\varepsilon > 0$,

$$\left|\operatorname{Im}\big(\partial\varphi(u), \partial\psi_\varepsilon(u)\big)\right| \leqslant c_q \operatorname{Re}\big(\partial\varphi(u), \partial\psi_\varepsilon(u)\big),$$

where $\partial\psi_\varepsilon$ is the Yosida approximation of $\partial\psi$.

Using the region $\mathrm{CGL}(y_0)$ introduced in Section 1 and the region $S(x_0)$ defined as

$$S(x_0) := \big\{(x, y) \in \mathbb{R}^2;\ |x| \leqslant x_0\big\} \quad (0 \leqslant x_0 < \infty), \qquad S(\infty) := \mathbb{R}^2, \tag{2.1}$$

we can state the existence theorem for global strong solutions to (ACP).

Theorem 2.1. *(See [16, Theorem 4.1].) Let $\lambda > 0$, $\kappa > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Assume that φ, ψ satisfy* (**A1**)–(**A5**) *and $(\alpha/\lambda, \beta/\kappa)$ satisfies*

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in S\left(\frac{1}{c_p}\right) \cap \mathrm{CGL}\left(\frac{1}{c_q}\right). \tag{2.2}$$

Then for any $u_0 \in D(\varphi) \cap D(\psi)$ there exists a global strong solution $u(\cdot) \in C([0,\infty); X)$ to (ACP) *such that*

(a) *$u(\cdot) \in C^{0,1/2}([0,T]; X)$ for every $T > 0$,*
(b) *$(du/dt)(\cdot), \partial\varphi(u(\cdot)), \partial\psi(u(\cdot)) \in L^2(0,T; X)$ for every $T > 0$,*
(c) *$\varphi(u(\cdot))$ and $\psi(u(\cdot))$ are absolutely continuous on $[0,T]$ for every $T > 0$,*

with the estimates

$$\|u(t)\| \leqslant e^{\gamma t}\|u_0\|, \quad t > 0, \tag{2.3}$$

$$E\bigl(u(t)\bigr) + \eta \int_0^t \bigl(\delta^2 \bigl\|\partial\varphi\bigl(u(s)\bigr)\bigr\|^2 + \bigl\|\partial\psi\bigl(u(s)\bigr)\bigr\|^2\bigr)\,ds \leqslant e^{\gamma_+ + rt} E(u_0), \quad t > 0, \tag{2.4}$$

where $\gamma_+ := \max\{\gamma, 0\}$, $r := \max\{p, q\}$,

$$E(u) := \delta^2\varphi(u) + \psi(u),$$

and $\delta > 0$, $\eta > 0$ *are constants depending only on* $\lambda, \kappa, \alpha, \beta, q$.

3. Proofs of Theorems 1.1–1.3

Let $R \geqslant 0$. Assume that (1.1) is satisfied. To apply the abstract theorem in Section 2 let $X := L^2(\mathbb{R}^N)$ with inner product $(\cdot,\cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$. Then we define two proper lower semi-continuous convex functions φ and ψ on X as follows:

$$\varphi(u) := \begin{cases} \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|V_R^{1/2}u\|_{L^2}^2) & \text{if } u \in D(\varphi) := H^1(\mathbb{R}^N) \cap D(V_R^{1/2}), \\ \infty & \text{otherwise}, \end{cases}$$

$$\psi(u) := \begin{cases} \frac{1}{q}\|u\|_{L^q}^q & \text{if } u \in D(\psi) := L^q(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \\ \infty & \text{otherwise}, \end{cases}$$

where $V_R \in C^1(\mathbb{R}^N; [0,\infty))$ is defined as (1.3) and $D(V_R^{1/2}) := \{u \in X;\ V_R^{1/2}u \in X\}$ is the Hilbert space with inner product $(\cdot,\cdot)_{D(V_R^{1/2})} = (\cdot,\cdot)_{L^2} + (V_R^{1/2}\cdot, V_R^{1/2}\cdot)_{L^2}$. Then the subdifferentials of φ and ψ are respectively given by

$$\partial\varphi(u) = (-\Delta + V_R)u \quad \text{for } u \in D(\partial\varphi) = \bigl\{u \in D(\varphi);\ (-\Delta + V_R)u \in L^2\bigl(\mathbb{R}^N\bigr)\bigr\},$$
$$\partial\psi(u) = |u|^{q-2}u \quad \text{for } u \in D(\partial\psi) = L^{2(q-1)}\bigl(\mathbb{R}^N\bigr) \cap L^2\bigl(\mathbb{R}^N\bigr).$$

Therefore (CGL) is regarded as one of (ACP)s given in Section 2.

Actually, the formal expression $\partial\varphi = -\Delta + V_R$ makes sense as an operator sum in X.

Lemma 3.1. *Define two nonnegative selfadjoint operators A and B as follows*:

$$Au := -\Delta u, \quad u \in D(A) := H^2\bigl(\mathbb{R}^N\bigr),$$
$$Bu := V_R u, \quad u \in D(B) := \bigl\{u \in L^2\bigl(\mathbb{R}^N\bigr);\ V_R u \in L^2\bigl(\mathbb{R}^N\bigr)\bigr\}.$$

Then $A + B = -\Delta + V_R$ *with* $D(A + B) := D(A) \cap D(B)$ *is also nonnegative and selfadjoint in* X, *and consequently, one can conclude that* $\partial\varphi = A + B$, *with*

$$\|Au\|_{L^2}^2 + \|Bu\|_{L^2}^2 \leqslant \bigl\|(A + B)u\bigr\|_{L^2}^2 + 2\|u\|_{L^2}^2, \quad u \in D(A + B). \tag{3.1}$$

Moreover, $C_0^\infty(\mathbb{R}^N)$ *is a core for* $A + B$.

Proof. Since $\nabla V_R(x) = 2(|x| - R)|x|^{-1}x$ if $|x| > R$, it follows that $|\nabla V_R(x)|^2 = 4V_R(x)$ on $\mathbb{R}^N$. Therefore, as shown in Okazawa [12, Example 6.5], we have

$$\operatorname{Re}(Au, B_n u)_{L^2} \geqslant -\|u\|_{L^2}, \quad n \in \mathbb{N},\ u \in D(A), \tag{3.2}$$

where $B_n = B_n(x) := V_R(x)(1 + n^{-1}V_R(x))^{-1}$ is the Yosida approximation of B. This implies that $A + B$ is selfadjoint in X (see [12, Theorem 5.4]). Letting $n \to \infty$ in (3.2) with $u \in D(A + B)$, we see that

$$\|Au\|_{L^2}^2 + \|Bu\|_{L^2}^2 \leqslant \|Au\|_{L^2}^2 + 2\operatorname{Re}(Au, Bu)_{L^2} + \|Bu\|_{L^2}^2 + 2\|u\|_{L^2}^2.$$

This is nothing but (3.1). Since $V_R \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$, it follows that $C_0^\infty(\mathbb{R}^N)$ is a core for $A + B$. This is a consequence of Kato's inequality (see Reed–Simon [18, Theorem X.28]). □

To verify the conditions on φ and ψ (e.g., (**A1**) and (**A5**)) assumed in the abstract theorem in Section 2, we need two lemmas.

Lemma 3.2. *Let A and B be as in Lemma 3.1. Then there exists a constant $C_R > 0$ such that $C_R \to 0$ $(R \to \infty)$ and*

$$\left\|(A + B + \xi)^{-1}\right\| \leqslant (\xi + C_R)^{-1}, \quad \xi > -C_R. \tag{3.3}$$

Consequently, $(A + B)^{-1/2}$ is well-defined:

$$(A + B)^{-1/2} = \frac{1}{\pi}\int_0^\infty \xi^{-1/2}(A + B + \xi)^{-1}\,d\xi, \tag{3.4}$$

with $\|(A + B)^{-1/2}\| \leqslant C_R^{-1/2}$. Moreover, $(A + B)^{-1}$ is compact together with $(A + B)^{-1/2}$.

Proof. First let $u \in C_0^\infty(\mathbb{R}^N)$. Setting $C_R := (1 + 2R^2/N)^{-1}(N/2)$, we have

$$C_R\|u\|_{L^2}^2 \leqslant \|\nabla u\|_{L^2}^2 + \left\|V_R^{1/2}u\right\|_{L^2}^2. \tag{3.5}$$

Indeed, integration by parts and Schwarz's inequality give

$$N\|u\|_{L^2}^2 = -\int_{\mathbb{R}^N} x \cdot \nabla|u|^2\,dx \leqslant 2\left\||x|u\right\|_{L^2}\|\nabla u\|_{L^2}. \tag{3.6}$$

Noting that $|x| = R + (|x| - R) = R + V_R^{1/2}(x)$ if $|x| > R$, we see that $|x| \leqslant R + V_R^{1/2}(x)$ for every $x \in \mathbb{R}$, and hence

$$\left\||x|u\right\|_{L^2} \leqslant R\|u\|_{L^2} + \left\|V_R^{1/2}u\right\|_{L^2}.$$

Plugging this inequality into the right-hand side of (3.6) yields

$$\begin{aligned} N\|u\|_{L^2}^2 &\leqslant 2R\|u\|_{L^2}\|\nabla u\|_{L^2} + 2\|\nabla u\|_{L^2}\left\|V_R^{1/2}u\right\|_{L^2} \\ &\leqslant \frac{N}{2}\|u\|_{L^2}^2 + \left(\frac{2}{N}R^2 + 1\right)\|\nabla u\|_{L^2}^2 + \left\|V_R^{1/2}u\right\|_{L^2}^2. \end{aligned}$$

This shows (3.5). Noting $\|\nabla u\|_{L^2}^2 + \|V_R^{1/2}u\|_{L^2}^2 = ((A+B)u, u)_{L^2}$, we have

$$(\xi + C_R)\|u\|_{L^2}^2 \leqslant \big((A+B+\xi)u, u\big)_{L^2}, \quad \xi > -C_R.$$

Since $C_0^\infty(\mathbb{R}^N)$ is a core for $A+B$, this inequality holds for every $u \in D(A+B)$. Thus we obtain (3.3). Therefore $(A+B)^{-1/2}$ is well-defined and given by (3.4); note that the integral converges in norm:

$$\big\|(A+B)^{-1/2}\big\| \leqslant \frac{1}{\pi}\int_0^\infty \xi^{-1/2}(\xi + C_R)^{-1}\,d\xi = C_R^{-1/2}$$

by using a change of variable. Since $V_R(x) \to \infty$ as $|x| \to \infty$, it follows that $(A+B)^{-1}$ is compact (cf. Okazawa [13, Theorem 4.1]; see also Reed–Simon [19, Theorem XIII.67]). In view of (3.4) we see that $(A+B)^{-1/2}$ is also compact (see Kato [5, Theorem V.3.49]). $\square$

Lemma 3.3. *Let $q \geqslant 2$. Let A, B and ψ be as in Lemma* 3.1. *Then the following assertions hold*:

(**i**) *For $u \in D(A)$ and $\varepsilon > 0$,*

$$\big|\mathrm{Im}\big(Au, \partial\psi_\varepsilon(u)\big)_{L^2}\big| \leqslant \frac{q-2}{2\sqrt{q-1}}\,\mathrm{Re}\big(Au, \partial\psi_\varepsilon(u)\big)_{L^2},$$

where $\partial\psi_\varepsilon$ is defined in Section 2.

(**ii**) *For $u \in D(B)$ and $\varepsilon > 0$,*

$$\big(Bu, \partial\psi_\varepsilon(u)\big)_{L^2} = \int_{\mathbb{R}^N} V_R|u_\varepsilon|^q\,dx + \varepsilon\int_{\mathbb{R}^N} V_R|u_\varepsilon|^{2(q-1)}\,dx, \tag{3.7}$$

where $u_\varepsilon := (1+\varepsilon\partial\psi)^{-1}u$. Consequently, $(Bu, \partial\psi_\varepsilon(u))_{L^2}$ is real and nonnegative.

(**iii**) *The addition of* (**i**) *and* (**ii**) *yields that for $u \in D(A+B)$ and $\varepsilon > 0$,*

$$\big|\mathrm{Im}\big((A+B)u, \partial\psi_\varepsilon(u)\big)_{L^2}\big| \leqslant \frac{q-2}{2\sqrt{q-1}}\,\mathrm{Re}\big((A+B)u, \partial\psi_\varepsilon(u)\big)_{L^2}.$$

Proof. (**i**) is known (see [16, Lemma 6.2]). To prove (**ii**) put $u_\varepsilon := (1+\varepsilon\partial\psi)^{-1}u$ for $u \in D(B)$ and $\varepsilon > 0$. Then we have

$$u = u_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}u_\varepsilon, \qquad \partial\psi_\varepsilon(u) = |u_\varepsilon|^{q-2}u_\varepsilon.$$

Therefore (3.7) is a direct consequence of the identity $Bu = V_R(u_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}u_\varepsilon)$. Finally, (**iii**) follows from (**i**) and (**ii**). $\square$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $R \geqslant 0$. Let (1.1) be satisfied by the various constants. Under the above notation, we show that φ and ψ satisfy (**A1**)–(**A5**) in Section 2. First we note that since $\partial\varphi = A+B$ is linear, the following conditions hold by Lemma 3.1:

(**A2**)′ $\varphi(\zeta u) = |\zeta|^2\varphi(u)$, $u \in D(\varphi)$, $\zeta \in \mathbb{C}$, $\mathrm{Re}\,\zeta > 0$.
(**A4**)′ $((A+B)u, u) \geqslant 0$, $u \in D(\partial\varphi)$.

Noting that $\varphi(u) = (1/2)\|(A+B)^{1/2}u\|_{L^2}^2$ for $u \in D(\varphi)$, we have

$$\{u \in D(\varphi);\ \varphi(u) \leqslant c\} = \{(A+B)^{-1/2}v;\ v \in L^2(\mathbb{R}^N),\ \|v\|_{L^2}^2 \leqslant 2c\}.$$

Therefore (**A1**) follows from the compactness of $(A+B)^{-1/2}$ (see Lemma 3.2). Clearly condition (**A2**) with $p = 2$ is (**A2**)$'$, while condition (**A3**) with $q \geqslant 2$ follows from the definition of ψ. Then (**A4**) with $c_2 = 0$ is a direct consequence of (**A4**)$'$ and finally Lemma 3.3(**iii**) implies (**A5**) with

$$c_q = \frac{q-2}{2\sqrt{q-1}}.$$

So we can apply Theorem 2.1 with these φ and ψ. Consequently, under the conditions

$$u_0 \in D(\varphi) \cap D(\psi) = H^1(\mathbb{R}^N) \cap D\big(V_R^{1/2}\big) \cap L^q(\mathbb{R}^N), \tag{1.2}$$

$$\left(\frac{\alpha}{\lambda}, \frac{\beta}{\kappa}\right) \in S\left(\frac{1}{c_2}\right) \cap \mathrm{CGL}\left(\frac{1}{c_q}\right) = \mathrm{CGL}\left(\frac{2\sqrt{q-1}}{q-2}\right), \tag{1.4}$$

we can conclude that $(\mathrm{CGL})_R$ admits a global strong solution $u(\cdot) \in C([0,\infty); L^2(\mathbb{R}^N))$ in the sense of Definition 1.1. Indeed, condition (b) in Definition 1.1 is verified by the separation property (3.1). As in the proof of [16, Theorem 1.1], we can prove (1.7) by using Theorem 2.1(c). Finally, (1.8) and (1.9) follow from (2.3) and (2.4), respectively. □

Next we prove Theorem 1.2. The key lies in the fact that

$$V_R \to 0 \quad (R \to \infty) \quad \text{uniformly on bounded sets of } \mathbb{R}^N.$$

Proof of Theorem 1.2. Let (1.1) and (1.4) be satisfied. We divide the proof into two steps.

(Step 1) Let $u_0 \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$. Then we note that $u_0 \in D(V_R^{1/2})$ for every $R \geqslant 0$. Hence it follows from Theorem 1.1 that there exists a strong solution to $(\mathrm{CGL})_R$ with (1.7), (1.8) and (1.9). Though the uniqueness of strong solutions to $(\mathrm{CGL})_R$ cannot be guaranteed, we choose and denote by $u_R(\cdot)$ one of strong solutions to $(\mathrm{CGL})_R$ with estimates (1.8) and (1.9) for every $R \geqslant 0$. In view of (1.8) we see that there exist a sequence $\{u_{R_n}(\cdot)\}$ selected from $\{u_R(\cdot)\}$ and a function $u(\cdot) : (0,\infty) \to L^2(\mathbb{R}^N)$ such that

$$u_{R_n}(\cdot) \to u(\cdot) \quad (n \to \infty) \quad \text{weakly in } L^2\big(0,T; L^2(\mathbb{R}^N)\big) \quad \forall T > 0. \tag{3.8}$$

Now let $T > 0$. Then (1.9) together with (3.1) implies that $\{\Delta u_R(\cdot)\}_{R \geqslant 0}$, $\{V_R u_R(\cdot)\}_{R \geqslant 0}$, $\{|u_R|^{q-2}u_R(\cdot)\}_{R \geqslant 0}$ are bounded in $L^2(0,T; L^2(\mathbb{R}^N))$ and so is $\{\partial u_R(\cdot)/\partial t\}_{R \geqslant 0}$, too by the equation; note that $\{E_R(u_0)\}_{R \geqslant 0}$ is bounded:

$$E_R(u_0) \leqslant \frac{\delta^2}{2}\big[\|\nabla u_0\|_{L^2}^2 + \big\||x|u_0\big\|_{L^2}^2\big] + \frac{1}{q}\|u_0\|_{L^q}^q.$$

First we show that

$$V_{R_n} u_{R_n}(\cdot) \to 0 \quad (n \to \infty) \quad \text{weakly in } L^2\big(0,T; L^2(\mathbb{R}^N)\big). \tag{3.9}$$

To this end let $\varphi \in C_0^\infty(\mathbb{R}^N \times (0,T))$. Then for sufficiently large $R > 0$ the intersection of $\operatorname{supp} V_R$ and $\bigcup_{t \in (0,T)} \operatorname{supp} \varphi(t)$ is empty and hence

$$\int_0^T \bigl(V_R u_R(t), \varphi(t)\bigr)_{L^2}\, dt = \int_0^T \bigl(u_R(t), V_R\varphi(t)\bigr)_{L^2}\, dt = 0.$$

Since $C_0^\infty(\mathbb{R}^N \times (0,T))$ is dense in $L^2(0,T;L^2(\mathbb{R}^N))$, (3.9) follows from the boundedness of $\{V_R u_R(\cdot)\}_{R\geqslant 0}$.

Next we show that $u(\cdot) \in L^2(0,T;H^2(\mathbb{R}^N)) \cap H^1(0,T;L^2(\mathbb{R}^N))$ and

$$\Delta u_{R_n}(\cdot) \to \Delta u(\cdot) \quad (n \to \infty) \quad \text{weakly in } L^2\bigl(0,T;L^2\bigl(\mathbb{R}^N\bigr)\bigr), \tag{3.10}$$

$$\frac{\partial u_{R_n}}{\partial t}(\cdot) \to \frac{\partial u}{\partial t}(\cdot) \quad (n \to \infty) \quad \text{weakly in } L^2\bigl(0,T;L^2\bigl(\mathbb{R}^N\bigr)\bigr). \tag{3.11}$$

Indeed, these are consequences of (3.8) and the weak closedness of Δ and $\partial/\partial t$ as operators in $L^2(0,T;L^2(\mathbb{R}^N))$.

Finally we would like to conclude that

$$|u_{R_n}|^{q-2}u_{R_n}(\cdot) \to |u|^{q-2}u(\cdot) \quad (n \to \infty) \quad \text{weakly in } L^2\bigl(0,T;L^2\bigl(\mathbb{R}^N\bigr)\bigr). \tag{3.12}$$

Taking a subsequence of $\{u_{R_n}(\cdot)\}$ (if necessary), we see that there exists a function $w(\cdot) \in L^2(0,T;L^2(\mathbb{R}^N))$ such that

$$|u_{R_n}|^{q-2}u_{R_n}(\cdot) \to w(\cdot) \quad (n \to \infty) \quad \text{weakly in } L^2\bigl(0,T;L^2\bigl(\mathbb{R}^N\bigr)\bigr). \tag{3.13}$$

It remains to show that $w = |u|^{q-2}u$. Since $\{u_{R_n}(\cdot)\}$, $\{\nabla u_{R_n}(\cdot)\}$ and $\{\partial u_{R_n}(\cdot)/\partial t\}$ are bounded in $L^2(0,T;L^2(\mathbb{R}^N)) = L^2(\mathbb{R}^N \times (0,T))$ by virtue of (1.8) and (1.9), it follows that $\{u_{R_n}\}$ is bounded in $H^1(\mathbb{R}^N \times (0,T))$. Let $\Omega \subset \mathbb{R}^N$ be any bounded domain. Then $\{u_{R_n}\}$ is bounded in $H^1(\Omega \times (0,T))$, and hence Rellich's theorem implies that

$$u_{R_n} \to u \quad (n \to \infty) \quad \text{strongly in } L^2\bigl(\Omega \times (0,T)\bigr). \tag{3.14}$$

Setting $\mathcal{A}f := |f|^{q-2}f$ for $f \in D(\mathcal{A}) := L^{2(q-1)}(\Omega \times (0,T))$, we see that $\mathcal{A}$ is represented by a subdifferential operator such as $\partial\psi$ at the beginning of this section, and hence $\mathcal{A}$ is demiclosed in $L^2(\Omega \times (0,T))$ (see [1, Proposition 2.5, Exemple 2.3.4]). Therefore (3.13) and (3.14) yields that $w = \mathcal{A}u = |u|^{q-2}u$ a.e. on $\Omega \times (0,T)$. Since $\Omega \subset \mathbb{R}^N$ is arbitrary, we conclude that $w = |u|^{q-2}u$ a.e. on $\mathbb{R}^N \times (0,T)$.

Letting $n \to \infty$ in $(\mathrm{CGL})_{R_n}$, we see from (3.8)–(3.12) that $u(\cdot)$ is a strong solution to (CGL), with the estimates (1.8) and (1.11). Here we note that the initial condition in (CGL) is verified as follows. Setting $M := \sup_{n\in\mathbb{N}} \|\partial u_{R_n}(\cdot)/\partial t\|_{L^2(0,T;L^2(\mathbb{R}^N))}$, we have

$$\bigl\|u_{R_n}(t) - u_0\bigr\|_{L^2} \leqslant \int_0^t \left\|\frac{\partial u_{R_n}}{\partial t}(s)\right\|_{L^2} ds \leqslant M\sqrt{t}, \quad t \in [0,T].$$

Letting $n \to \infty$ gives $\|u(t) - u_0\|_{L^2} \leqslant M\sqrt{t}$. This shows that $u(t) \to u_0$ $(t \downarrow 0)$ in $L^2(\mathbb{R}^N)$. Finally, (1.10) is shown in the same way as in the proof of (1.7).

(Step 2) Let $u_0 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ as in (1.6). Setting

$$u_{0n} := \bigl(1 + n^{-1}|x|\bigr)^{-1}u_0, \quad n \in \mathbb{N},$$

we observe that $u_{0n} \in H^1(\mathbb{R}^N) \cap D(|x|) \cap L^q(\mathbb{R}^N)$ with

$$|u_{0n}| \leqslant |u_0|, \qquad \big||x|u_{0n}\big| \leqslant n|u_0|, \qquad |\nabla u_{0n}| \leqslant |\nabla u_0| + n^{-1}|u_0|.$$

Moreover, it follows that $u_{0n} \to u_0$ $(n \to \infty)$ both in $H^1(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$. Hence **(Step 1)** shows that there exists a strong solution $u_n(\cdot)$ to (CGL), with the estimates

$$\big\|u_n(t)\big\|_{L^2} \leqslant e^{\gamma t}\|u_{0n}\|_{L^2},$$

$$E_\infty\big(u_n(t)\big) + \eta \int_0^t \big\{\delta^2 \big\|\Delta u_n(s)\big\|_{L^2}^2 + \big\|u_n(s)\big\|_{L^{2(q-1)}}^{2(q-1)}\big\}\,ds \leqslant e^{\gamma_+ qt} E_\infty(u_{0n}).$$

Since $\{\|u_{0n}\|_{L^2}\}_{n\in\mathbb{N}}$ and $\{E_\infty(u_{0n})\}_{n\in\mathbb{N}}$ are bounded, the same argument as in **(Step 1)** yields that there exist a subsequence $\{u_{n_k}(\cdot)\}$ and a function $u(\cdot) : (0,\infty) \to L^2(\mathbb{R}^N)$ such that for every $T > 0$,

$$\begin{aligned}
u_{n_k}(\cdot) \to u(\cdot) \quad (k \to \infty) \quad &\text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big),\\
\Delta u_{n_k}(\cdot) \to \Delta u(\cdot) \quad (k \to \infty) \quad &\text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big),\\
\frac{\partial u_{n_k}}{\partial t}(\cdot) \to \frac{\partial u}{\partial t}(\cdot) \quad (k \to \infty) \quad &\text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big),\\
|u_{n_k}|^{q-2}u_{n_k}(\cdot) \to |u|^{q-2}u(\cdot) \quad (k \to \infty) \quad &\text{weakly in } L^2\big(0,T; L^2\big(\mathbb{R}^N\big)\big).
\end{aligned}$$

Therefore we conclude that $u(\cdot)$ is a strong solution to (CGL) and (1.8), (1.10) and (1.11) hold; the initial condition can be verified in the same way as in **(Step 1)**. This completes the proof. $\square$

To prove the assertion for (CGL) in Theorem 1.3, we can use the existence and uniqueness theorem [17, Theorem 3.1]; however, since the existence part has already been established in Theorems 1.1 and 1.2, we give a direct and simplified proof of the uniqueness.

Proof of Theorem 1.3. Let $R \geqslant 0$. We prove the assertion only for $(\mathrm{CGL})_R$. Concerning (CGL), we can modify the following proof with (1.2), V_R, $E_R(u_0)$, $E_R(v_0)$ replaced with (1.6), $V_R \equiv 0$, $E_\infty(u_0)$, $E_\infty(v_0)$, respectively. Assume that (1.1), (1.2), (1.4), (1.12) are satisfied; and so $q < 2^*$. Then it suffices to prove (1.13) which implies the uniqueness of strong solutions. Let $u(\cdot)$ and $v(\cdot)$ be strong solutions to $(\mathrm{CGL})_R$ with respective initial data u_0, v_0 satisfying (1.2). Then $w(\cdot) := u(\cdot) - v(\cdot)$ satisfies

$$\frac{\partial w}{\partial t} + (\lambda + i\alpha)(-\Delta + V_R)w + (\kappa + i\beta)\big(|u|^{q-2}u - |v|^{q-2}v\big) = \gamma w.$$

Hence we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \lambda\big(\|\nabla w\|_{L^2}^2 + \big\|V_R^{1/2}w\big\|_{L^2}^2\big) + I(u,v) = \gamma\|w\|_{L^2}^2, \tag{3.15}$$

where

$$I(u,v) := \mathrm{Re}\big[(\kappa + i\beta)\big(|u|^{q-2}u - |v|^{q-2}v, w\big)_{L^2}\big].$$

As in the proof of [17, Lemma 4.5], we see from Hölder's, Gagliardo–Nirenberg's and Young's inequalities that

$$\begin{aligned}
\left|I(u,v)\right| &\leqslant k_1\left(\frac{1}{2}\|u\|_{L^q}^q+\frac{1}{2}\|v\|_{L^q}^q\right)^{(q-2)/q}\|w\|_{L^q}^2 \\
&\leqslant \frac{\lambda}{2}\|\nabla w\|_{L^2}^2+k_2\left(\frac{1}{2}\|u\|_{L^q}^q+\frac{1}{2}\|v\|_{L^q}^q\right)^{\theta}\|w\|_{L^2}^2,
\end{aligned} \tag{3.16}$$

where $\theta := \frac{2(q-2)}{2N-(N-2)q} \in [0,\infty)$, $k_1 = k_1(q,\kappa,\beta) > 0$ and $k_2 = k_2(\lambda, N, q, \kappa, \beta) > 0$ are constants. In view of (1.9) we have

$$\frac{1}{2}\left\|u(t)\right\|_{L^q}^q+\frac{1}{2}\left\|v(t)\right\|_{L^q}^q \leqslant qe^{\gamma+qt}\max\left\{E_R(u_0), E_R(v_0)\right\}. \tag{3.17}$$

Plugging (3.16) and (3.17) into (3.15) yields

$$\frac{d}{dt}\|w\|_{L^2}^2+\lambda\left(\|\nabla w\|_{L^2}^2+\left\|V_R^{1/2}w\right\|_{L^2}^2\right)\leqslant K(t)\|w\|_{L^2}^2,$$

where

$$K(t) := 2\gamma+2k_2q^{\theta}e^{\gamma+q\theta t}\max\left\{E_R(u_0)^{\theta}, E_R(v_0)^{\theta}\right\}.$$

Therefore it follows that

$$\frac{d}{ds}\left[e^{\int_s^t K(r)\,dr}\left\|w(s)\right\|_{L^2}^2\right]+\lambda e^{\int_s^t K(r)\,dr}\left(\left\|\nabla w(s)\right\|_{L^2}^2+\left\|V_R^{1/2}w(s)\right\|_{L^2}^2\right)\leqslant 0.$$

Integrating this inequality over $[0,t]$ for $t>0$, we obtain the desired inequality (1.13). $\square$

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